On Formulas for the Velocity of Rayleigh Waves in Prestrained Incompressible Elastic Solids

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In the present paper, formulas for the velocity of Rayleigh waves in incompressible isotropic solids subject to a general pure homogeneous prestrain are derived using the theory of cubic equation. They have simple algebraic form and hold for a general strain-energy function. The formulas are concretized for some specific forms of strain-energy function. They then become totally explicit in terms of parameters characterizing the material and the prestrains. These formulas recover the (exact) value of the dimensionless speed of Rayleigh wave in incompressible isotropic elastic materials (without prestrain). Interestingly that, for the case of hydrostatic stress, the formula for the Rayleigh wave velocity does not depend on the type of strain-energy function.
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1 Introduction

Elastic surface waves in isotropic elastic solids, discovered by Lord Rayleigh [1] more than 120 years ago, have been studied extensively and exploited in a wide range of applications in seismology, acoustics, geophysics, telecommunication industry, and materials science, for example. It would not be far-fetched to say that Rayleigh's study of surface waves upon an elastic half-space has had fundamental and far-reaching effects on modern life and many things that we take for granted today, stretching from mobile phones through to the study of earthquakes, as stressed by Adams et al. [2].

For the Rayleigh wave, its speed is a fundamental quantity, which interests researchers in seismology and geophysics, and in other fields of physics and the material sciences. It is discussed in almost every survey and monograph on the subject of surface acoustic waves in solids. Furthermore, it also involves Green's function for many elastodynamic problems for a half-space, explicit formulas for the Rayleigh wave speed are clearly of practical, as well as theoretical interest.

In 1995, a first formula for the Rayleigh wave speed in compressible isotropic elastic solids have been obtained by Rahman and Barber [3], but for a limited range of values of the parameter $\epsilon=\mu/(\lambda+2\mu)$, where λ and μ are the usual Lame constants, by using the theory of cubic equations. Employing Riemann problem theory Nkemzi [4] derived a formula for the velocity of Rayleigh waves expressed as a continuous function of ϵ for any range of values. It is rather cumbersome [5], and the final result, as printed in his paper, is incorrect [6]. Malischewsky [6] obtained a formula for the speed of Rayleigh waves for any range of values of ϵ by using Cardan's formula together with trigonometric formulas for the roots of a cubic equation and MATHEMATICA. It is expressed as a continuous function of ϵ. In Malischewsky's paper [6] it is not shown, however, how Cardan's formula together with the trigonometric formulas for the roots of the cubic equation are used with MATHEMATICA to obtain the formula. A detailed derivation of this formula was given by Pham and Ogden [7] together with an alternative formula. For nonisotropic materials, for some special cases of compressible monoclinic materials with symmetry plane, formulas for the Rayleigh wave speed have been found by Ting [8] and Destrade [5] as the roots of quadratic equations, while for incompressible orthotropic materials an explicit formula has been given by Ogden and Pham [9] based on the theory of cubic equations. Furthermore, in recent papers [10,11] Pham and Ogden have obtained explicit formulas for the Rayleigh wave speed in compressible orthotropic elastic solids.

Nowaday prestressed materials have been widely used. Nondestructive evaluation of prestresses of structures before and during loading (in the course of use) becomes necessary and important, and the Rayleigh wave is a convenient tool for this task, see, for example, Refs. [12–15]. In these studies (also in Refs. [16,17]), for evaluating prestresses by the Rayleigh wave, the authors have established the (approximate) formulas for the relative variation in the Rayleigh wave velocity [12,15] or its variation [16,17]). They are linear in terms of the prestrains (or prestresses), thus, they are very convenient to use. However, since these formulas are derived by using the perturbation method they are only valid for enough small prestrains. They are no longer to be applicable when prestrains are not small.

The main purpose of this paper is to find (exact) formulas for the velocity of Rayleigh waves in incompressible isotropic elastic materials subject to a general pure homogeneous prestrain by using the theory of cubic equation. Since they are valid for any range of prestrain, they will provide a powerful tool for the nondestructive evaluation of prestresses of structures.

The paper is organized as follows. The derivation of the secular equation of Rayleigh waves in a half-space of incompressible isotropic material subject to a generally pure homogeneous prestrain is presented briefly in Sec. 2. The formulas for the Rayleigh wave velocity are derived in Sec. 3. In this section the necessary and sufficient conditions for the unique existence of the dimensionless Rayleigh wave speed x_r are also established. In Sec. 4, concretization of formulas is carried out for a number of particular strain-energy functions, and the obtained formulas are then totally explicit with respect to the parameters characterizing the material and the prestrains. It is noted that, for the case of hydrostatic stress, the formula for Rayleigh wave velocity does not depend on the type of strain-energy function.

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2 Secular Equation

In this section we first summarize the basic equations, which govern small amplitude time-dependent motions superimposed upon a large static primary deformation, under the assumption of incompressibility, and then present briefly the derivation of the secular equation of Rayleigh waves in prestrained elastic solids. For more details, the reader is referred to the paper by Dowaikh and Ogden [18].

We consider an unstressed body corresponding to the half-space $X_2 \le 0$ and we suppose that the deformed configuration is obtained by application of a pure homogeneous strain of the form

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \quad \lambda_i = \text{const}, \quad i = 1,2,3 \tag{1}$$

where $\lambda_i > 0$, $i = 1,2,3$ are the principal stretches of the deformation. In its deformed configuration the body, therefore, occupies the region $x_2 < 0$ with the boundary $x_2 = 0$.

We consider a plane motion in the (x_1, x_2)-plane with displacement components u_1, u_2, and u_3 such that

$$u_i = u_i(x_1, x_2, t), \quad i = 1,2, \quad u_3 \equiv 0 \tag{2}$$

where t is the time. Then in the absence of body forces the equations governing infinitesimal motion, expressed in terms of displacement components u_i, are

$$B_{1111}u_{1,11} + (B_{1122} + B_{2112})u_{2,21} + B_{2121}u_{1,22} - p^*_{,1} = \rho\ddot{u}_1$$

$$(B_{1221} + B_{2211})u_{1,12} + B_{1212}u_{2,11} + B_{2222}u_{2,22} - p^*_{,2} = \rho\ddot{u}_2 \tag{3}$$

where p^* is a time-dependent pressure increment, ρ is mass density of the material, a superposed dot signifies differentiation with respect to t, commas indicate differentiation with respect to spatial variables x_i, B_{ijkl} is a component of the fourth order elasticity tensor defined as follows:

$$B_{iijj} = \lambda_i \lambda_j \frac{\partial^2 W}{\partial \lambda_i \, \partial \lambda_j} \tag{4}$$

$$B_{ijij} = \begin{cases} \left(\lambda_i \dfrac{\partial W}{\partial \lambda_i} - \lambda_j \dfrac{\partial W}{\partial \lambda_j}\right) \dfrac{\lambda_i^2}{\lambda_i^2 - \lambda_j^2}, & (i \neq j, \lambda_i \neq \lambda_j) \\ \dfrac{1}{2}\left(B_{iiii} - B_{iijj} + \lambda_i \dfrac{\partial W}{\partial \lambda_i}\right) & (i \neq j, \lambda_i = \lambda_j) \end{cases} \tag{5}$$

$$B_{ijji} = B_{jiij} = B_{ijij} - \lambda_i \frac{\partial W}{\partial \lambda_i} (i \neq j) \tag{6}$$

for $i,j \in 1,2,3$, $W = W(\lambda_1, \lambda_2, \lambda_3)$ (noting that $\lambda_1\lambda_2\lambda_3 = 1$) is the strain-energy function per unit volume, all other components being zero. In the stress-free configuration, Eqs. (4)–(6) reduce to

$$B_{iiii} = B_{ijij} = \mu (i \neq j), \quad B_{iijj} = B_{ijji} = 0 (i \neq j) \tag{7}$$

where μ is the shear modulus of the material in that configuration.

Equation of motion (3) are taken together with the boundary conditions of zero incremental traction, which are expressed as

$$B_{2121}u_{1,2} + (B_{2121} - \sigma_2)u_{2,1} = 0 \quad \text{on} \quad x_2 = 0$$

$$(B_{1122} - B_{2222} - p)u_{1,1} - p^* = 0 \quad \text{on} \quad x_2 = 0 \tag{8}$$

where p denotes a static pressure in the prestressed equilibrium state, $\sigma_i (i = 1,2,3)$ are the principal Cauchy stresses given by

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \tag{9}$$

For an incompressible material, we have

$$u_{1,1} + u_{2,2} = 0 \tag{10}$$

From Eq. (10) we deduce the existence of a function ψ of x_1, x_2, and t such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1} \tag{11}$$

Elimination of p^* from Eq. (3) and use of Eq. (11) then yield an equation for ψ having the form

$$\alpha\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}) \tag{12}$$

where

$$\alpha = B_{1212}, \quad \gamma = B_{2121}, \quad 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221} \tag{13}$$

It is noted from the strong-ellipticity condition of system (3) that α, β, and γ are required to satisfy the inequalities

$$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma} \tag{14}$$

Differentiation of the second of Eq. (8) with respect to x_1 followed by use of Eqs. (3), (9), and (11) then allows Eq. (8) to be put in the form

$$\gamma(\psi_{,22} - \psi_{,11}) + \sigma_2\psi_{,11} = 0 \quad \text{on} \quad x_2 = 0$$

$$(2\beta + \gamma - \sigma_2)\psi_{,112} + \gamma\psi_{,222} - \rho\ddot{\psi}_{,2} = 0 \quad \text{on} \quad x_2 = 0 \tag{15}$$

We now consider a wave propagating in the x_1-direction. For this to be a surface wave, the displacement u_1 and u_2 and, hence, ψ must decay to zero as $x_2 \to -\infty$. We therefore take ψ to have the form

$$\psi = (Ae^{ks_1x_2} + Be^{ks_2x_2})\exp(kx_1 - \omega t) \tag{16}$$

where A and B are constants, ω is the wave frequency, k is the wavenumber, c is the wave speed, and s_1 and s_2 are roots of the equation

$$\gamma s^4 - (2\beta - \rho c^2)s^2 + \alpha - \rho c^2 = 0 \tag{17}$$

From Eq. (17) we have

$$s_1^2 + s_2^2 = (2\beta - \rho c^2)/\gamma, \quad s_1^2 s_2^2 = (\alpha - \rho c^2)/\gamma \tag{18}$$

For decay of ψ as $x_2 \to -\infty$, s_1, s_2 are required to have positive real parts. The roots s_1^2 and s_2^2 of the quadratic Eq. (17) are either both real (and, if so, both positive because of positive real parts of s_1 and s_2) or they are a complex conjugate pair. In either case: $s_1^2 s_2^2 > 0$ and so, by Eq. (18) and $\gamma > 0$

$$0 < \rho c^2 < \alpha \tag{19}$$

Substitution of Eq. (16) into the boundary condition (15) yields

$$(\gamma s_1^2 + \gamma - \sigma_2)A + (\gamma s_2^2 + \gamma - \sigma_2)B = 0$$

$$(2\beta + \gamma - \sigma_2 - \rho c^2 - \gamma s_1^2)s_1 A + (2\beta + \gamma - \sigma_2 - \rho c^2 - \gamma s_2^2)s_2 B = 0 \tag{20}$$

For nontrivial solution of Eq. (20) for A and B, the determinant of coefficients must vanish. After some algebra and after using Eq. (18), removal of a factor $s_1 - s_2$ leads to

$$\gamma(\alpha - \rho c^2) + (2\beta + 2\gamma - 2\sigma_2 - \rho c^2)[\gamma(\alpha - \rho c^2)]^{1/2} = \gamma_*^2 \tag{21}$$

where $\gamma_* = \gamma - \sigma_2$. It is noted that vanishing of the factor $s_1 - s_2$ yields a trivial solution. Equation (21) is the secular equation, which determines the speed c of propagation of surface (Rayleigh) waves of the type considered. It follows from Eq. (19) that the Rayleigh wave speed c has to satisfy the inequalities

$$0 < c^2 < c_2^2 = \alpha/\rho \tag{22}$$

3 Formulas for the Rayleigh Wave Velocity in Prestrained Incompressible Solids

In order to proceed it is convenient to introduce three dimensionless parameters defined as follows:

$$\delta_1 = \gamma/\alpha, \quad \delta_2 = \beta/\alpha, \quad \delta_3 = \gamma_*/\alpha \tag{23}$$

It is noted from Eqs. (14) and (23) that

$$\delta_1 > 0 \quad (24)$$

We also define the variable x by

$$x = c^2/c_2^2 \quad (25)$$

and in terms of the new variable x, Eq. (21) is of the form

$$\delta_1(1-x) + \sqrt{\delta_1}(2\delta_2 + 2\delta_3 - x)\sqrt{1-x} = \delta_3^2 \quad (26)$$

It follows from Eqs. (22) and (25)

$$0 < x < 1 \quad (27)$$

On introducing the variable t given by

$$t = \sqrt{1-x} \quad (28)$$

Eq. (26) becomes

$$F(t) \equiv t^3 + \sqrt{\delta_1}t^2 + (2\delta_2 + 2\delta_3 - 1)t - \frac{\delta_3^2}{\sqrt{\delta_1}} = 0 \quad (29)$$

It follows from Eqs. (27) and (28)

$$0 < t < 1 \quad (30)$$

It is noted that Eq. (28) is a 1-1 mapping from (0,1) to itself. From Eq. (29) it follows

$$F'(t) = 3t^2 + 2\sqrt{\delta_1}t + 2\delta_2 + 2\delta_3 - 1 \quad (31)$$

It is clear from Eqs. (24) and (31) that if the equation $F'(t)=0$ has two distinct real roots, denoted by $t_{\max}$, $t_{\min}(t_{\max}<t_{\min})$, then

$$t_{\max} + t_{\min} = -2\sqrt{\delta_1}/3 < 0 \quad (32)$$

Since the cubic equation (29) will be equivalently reduced to a quadratic one, in the interval (0,1) when $\delta_3=0$, we examine separately the cases: $\delta_3=0$ and $\delta_3 \neq 0$. By x_r we denote the solution of Eq. (26) satisfying Eq. (27), and call it the dimensionless Rayleigh wave velocity.

For Case 1 $\delta_3=0$, we have the following proposition.

PROPOSITION 1.

(i) *Suppose* $\delta_3=0$, then *Eq. (29) has a unique root in the interval (0,1) if and only if*

$$-\sqrt{\delta_1} < 2\delta_2 < 1 \quad (33)$$

(ii) *Let* $\delta_3=0$ and Eq. (33) *holds, then the dimensionless Rayleigh wave velocity is defined by following formula:*

$$x_r = [4 - (\sqrt{\delta_1 - 8\delta_2 + 4} - \sqrt{\delta_1})^2]/4 \quad (34)$$

Proof.

(i) Let $\delta_3=0$, then Eq. (29), in the interval (0,1), is equivalent to

$$\phi(t) \equiv t^2 + \sqrt{\delta_1}t + 2\delta_2 - 1 = 0 \quad (35)$$

Note that the coefficient of t^2 in the expression for $\phi(t)$ is a positive number.

- "$\Leftarrow$:" suppose Eq. (33) holds, then we have

$$\phi(0) < 0, \quad \phi(1) > 0 \quad (36)$$

It follows from the first part of Eq. (36) that Eq. (35) has two distinct real roots t_1 and t_2 and

$$t_1 < 0 < t_2 \quad (37)$$

By the second part of Eq. (36) we deduce

$$t_2 < 1 \quad (38)$$

From Eqs. (37) and (38) we conclude that Eq. (35), thus, Eq. (29) has a unique solution in the interval (0,1).

- "$\Rightarrow$:" suppose that Eq. (29), thus, Eq. (35) has a unique solution, namely, t_2 in the interval (0,1), then Eq. (35) has two distinct real roots t_1, t_2, otherwise

$$2t_2 = -\sqrt{\delta_1} < 0 \quad (39)$$

But this is impossible because $t_2>0$. Since two (real) roots of Eq. (35) are related by

$$t_1 + t_2 = -\sqrt{\delta_1} < 0 \quad (40)$$

we have (noting that $t_2>0$)

$$t_1 < 0 < t_2 \quad (41)$$

From Eq. (41) and noting that the coefficient of t^2 in the expression for $\phi(t)$ is a positive number, it follows

$$\phi(0) < 0, \quad \phi(1) > 0 \quad (42)$$

From Eq. (42) we obtain Eq. (33) and the proof of (i) is finished.

(ii) Suppose $\delta_3=0$ and Eq. (33) holds. According to (i), in this case, Eq. (29) has only one root denoted by t_r, in the interval (0,1). By the proof of (i), $t_r=t_2$ the bigger root of Eq. (35), thus, it is defined by

$$t_r = \frac{\sqrt{\delta_1 - 8\delta_2 + 4} - \sqrt{\delta_1}}{2} \quad (43)$$

From Eq. (28)

$$x_r = 1 - t_r^2 \quad (44)$$

and Eq. (34) is deduced from Eqs. (43) and (44).

Case 2. $\delta_3 \neq 0$.

PROPOSITION 2. *Suppose* $\delta_3 \neq 0$, then *Eq. (29) has a unique root in the interval (0,1) if and only if*

$$F(1) = \sqrt{\delta_1} + 2\delta_2 + 2\delta_3 - \frac{\delta_3^2}{\sqrt{\delta_1}} > 0 \quad (45)$$

Proof. Let $\delta_3 \neq 0$, it follows from Eq. (29) that $F(0)<0$.

If $d=2\delta_2+2\delta_3-1 \geq 0$, then from Eq. (31) $F'(t)>0$ for $\forall t>0$. Thus Eq. (29) has a unique root in the interval (0,1) if and only $F(1)>0$.

If $d<0$, then the equation $F'(t)=0$ has two distinct (real) roots $t_{\max}$, $t_{\min}$, and $t_{\max}<0<t_{\min}$. Since $F(0)<0$ and $F(t)$ is strictly decreasingly monotonous in $(t_{\max}, t_{\min})$, it follows that $F(t) < 0 \,\forall\, t \in (0, t_{\min}]$, i.e., the equation $F(t)=0$ has no root in the interval $(0, t_{\min}]$. Since $F(t)$ is strictly increasingly monotonous in the interval $(t_{\min}, +\infty)$, it is strictly increasingly monotonous in the interval $(t_{\min}, 1)$. This and $F(t_{\min})<0$ yield that Eq. (29) has a unique solution in the interval (0,1) if and only $F(1)>0$. The proof is completed.

Remark 1.

(i) Inequality (45) is equivalent to (6.9) in Ref. [18], namely

$$(\alpha - \gamma)\gamma + 2\sqrt{\alpha\gamma} + 2\sigma_2(\gamma - \sqrt{\alpha\gamma}) - \sigma_2^2 > 0 \quad (46)$$

and it gives

$$\gamma - \sqrt{\alpha\gamma} - \sqrt{2\sqrt{\alpha\gamma}(\beta + \sqrt{\alpha\gamma})} < \sigma_2 < \gamma - \sqrt{\alpha\gamma} + \sqrt{2\sqrt{\alpha\gamma}(\beta + \sqrt{\alpha\gamma})} \quad (47)$$

(ii) Inequality (33) is equivalent to (6.17) in Ref. [18] without the left-hand equality.

From the proof of the Proposition 2, we have immediately the following proposition.

PROPOSITION 3. *Suppose* $\delta_3 \neq 0$ and $F(1)>0$. If *Eq. (29) has two or three distinct real roots, then the root corresponding to the Rayleigh wave, say* t_r, *is the largest root*.

By introducing the notations

$$a_0 = -\frac{\delta_3^2}{\sqrt{\delta_1}}, \quad a_1 = 2\delta_2 + 2\delta_3 - 1, \quad a_2 = \sqrt{\delta_1} \tag{48}$$

Eq. (29) becomes

$$F(t) \equiv t^3 + a_2 t^2 + a_1 t + a_0 = 0 \tag{49}$$

In terms of the variable z given by

$$z = t + \tfrac{1}{3}a_2 \tag{50}$$

Eq. (49) has the form

$$z^3 - 3q^2 z + r = 0 \tag{51}$$

where

$$q^2 = (a_2^2 - 3a_1)/9, \quad r = (2a_2^3 - 9a_1 a_2 + 27a_0)/27 \tag{52}$$

It should be noted that here q^2 can be negative.

Our task is now to find the real solution z_r of Eq. (51), that is related to t_r by the relation (50). As t_r is the largest root of Eq. (49), z_r is the largest one in Eq. (51) in the case that it has two or three distinct real roots. By theory of cubic equation, three roots of Eq. (51) are given by the Cardan's formula as follows (see Ref. [19]):

$$z_1 = S + T$$

$$z_2 = -\tfrac{1}{2}(S+T) + \tfrac{1}{2}i\sqrt{3}(S-T)$$

$$z_3 = -\tfrac{1}{2}(S+T) - \tfrac{1}{2}i\sqrt{3}(S-T) \tag{53}$$

where $i^2=-1$ and

$$S = \sqrt[3]{R+\sqrt{D}}, \quad T = \sqrt[3]{R-\sqrt{D}}$$

$$D = R^2 + Q^3, \quad R = -\tfrac{1}{2}r, \quad Q = -q^2 \tag{54}$$

Remark 2. In relation to these formulas we emphasize two points:

(i) the cubic root of a real negative number is taken as the negative real root.

(ii) if the argument in S is complex we take the phase angle in T as the negative of the phase angle in S, such as $T=S^*$, where S^* is the complex conjugate value of S.

Remark 3. The nature of three roots of Eq. (51) depends on the sign of its discriminant D, in particular: If $D>0$, then Eq. (51) has one real root and two complex conjugate roots; if $D=0$, the equation has three real roots, at least two of which are equal; if $D<0$, then it has three real distinct roots.

We now show that in each case the largest real root of Eq. (51) z_r is given by

$$z_r = \sqrt[3]{R+\sqrt{D}} + \frac{q^2}{\sqrt[3]{R+\sqrt{D}}} \tag{55}$$

in which each radical is understood as complex roots taking its principle value, and

$$R = (9a_1 a_2 - 27a_0 - 2a_2^3)/54$$

$$D = (4a_0 a_2^3 - a_1^2 a_2^2 - 18a_0 a_1 a_2 + 27a_0^2 + 4a_1^3)/108 \tag{56}$$

where $a_i, i=0,1,2$ expressed in terms of the dimensionless parameters δ_i, $i=1,2,3$ by Eq. (48).

First, we note that one can obtain Eq. (56) by substituting Eq. (52) into Eq. (54). Now we examine the distinct cases dependent on the values of q^2 in order to prove Eq. (55).

Case 1. $q^2<0$.

If $q^2<0 \Rightarrow Q>0 \Rightarrow D=R^2+Q^3>0$. This ensures that

(i) $\sqrt{D}>|R| \Rightarrow \sqrt{D}+R>0$.

(ii) Eq. (51), by the Remark 3, has a unique real root, so it is z_r is given by the first of Eq. (53), in particular

$$z_r = \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}} \tag{57}$$

here the radicals are understood as real ones. Since

$$\sqrt[3]{R-\sqrt{D}}\,.\,\sqrt[3]{R+\sqrt{D}} = \sqrt[3]{R^2-D} = \sqrt[3]{(-Q)^3} = -Q = q^2 \tag{58}$$

we have

$$\sqrt[3]{R-\sqrt{D}} = \frac{q^2}{\sqrt[3]{R+\sqrt{D}}} \tag{59}$$

In view of Eqs. (57) and (59), $R+\sqrt{D}>0$ and the fact that for a positive real number, its real cubic root and complex cubic root taking its principal value are the same, the validity of Eq. (55) is clear.

Case 2. $q^2=0$.

When $q^2=0$, $F'(t)\geq 0\, \forall\, t\in(-\infty,+\infty)$, so function $F(t)$ is strictly increasingly monotonous $(-\infty,+\infty)$. Since $a_2=\sqrt{\delta_1}>0 \Rightarrow -a_2/3<0 \Rightarrow r=F(-a_2/3)<F(0)\leq 0 \Rightarrow r<0 \Rightarrow R>0$. In view of $q^2=0$, Eq. (51) has a unique real $\sqrt[3]{2R}$, so $z_r=\sqrt[3]{2R}$. In other hand, $q^2=0 \Rightarrow Q=0 \Rightarrow D=R^2 \Rightarrow R=+\sqrt{D}$. Using this and $q^2=0$, from Eq. (55) we have: $z_r=\sqrt[3]{2R}$. Thus, formula (55) is valid for this case.

Case 3. $q^2>0$.

We recall that in this case function $F(t)$ attains maximum and minimum values at $t_{\max}$ and $t_{\min}(t_{\max}<t_{\min})$, and they are subjected to Eq. (32).

(i) If $D>0$, then Eq. (51), according to the Remark 3, has a unique real root, so it is z_r given by Eq. (57) in which the radicals are understood as real ones. The use of Eq. (32) and the fact $r=F(-a_2/3)$, it is not difficult to prove that $r<0$ or equivalently $R>0$. This leads to: $R+\sqrt{D}>0$. In view of this inequality and Eq. (59), formula (57) coincides with Eq. (55). This means formula (55) is true.

(ii) If $D=0$, analogously as above, it is not difficult to observe that $r<0$, or equivalently, $R>0$. When $D=0$ we have $R^2=-Q^3=|q|^6 \Rightarrow R=|q|^3 \Rightarrow r=-2R=-2|q|^3$, so Eq. (51) becomes

$$z^3 - 3|q|z^2 - 2|q|^3 = 0 \tag{60}$$

whose roots are $z_1=2|q|$ and $z_2=-|q|$ (double root). This yields $z_r=2|q|$ according to the Proposition 3. From Eq. (55) and taking into account $q^2>0$, $D=0$, it follows $z_r=2|q|$. This shows the validity of Eq. (55).

(iii) If $D<0$, then Eq. (51) has three distinct real roots, and according to Proposition 3, z_r is the largest root. By arguments presented in Ref. [10] (p. 255) one can show that, in this case, the largest root z_r of Eq. (51) is given by

$$z_r = \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}} \tag{61}$$

within which each radical is understood as the complex root taking its principal value. By $\theta(\in(0\ ,\pi)$ we denote the phase angle of the complex number $R+i\sqrt{-D}$. It is not difficult to verify that

$$\sqrt[3]{R+\sqrt{D}} = |q|e^{i\theta}, \quad \sqrt{R-\sqrt{D}} = |q|e^{-i\theta} \tag{62}$$

where each radical is understood as the complex root taking its principal value. It follows from Eq. (62) that

$$\sqrt[3]{R-\sqrt{D}} = \frac{q^2}{\sqrt[3]{R+\sqrt{D}}} \tag{63}$$

By substituting Eq. (63) into Eq. (61) we obtain Eq. (55), and the validity of Eq. (55) is proved.

We are now in the position to state the following proposition.

PROPOSITION 4. *Suppose* $\delta_3 \neq 0$ *and Eq. (45) holds. Then, the dimensionless velocity* x_r *of Rayleigh waves in prestrained incompressible solids is given by*

$$x_r = 1 - \left[\sqrt[3]{R + \sqrt{D}} + \frac{q^2}{\sqrt[3]{R + \sqrt{D}}} - \frac{1}{3}a_2 \right]^2 \tag{64}$$

in which each radical is understood as complex roots taking its principle value, q^2*, and D and R are given by Eqs. (52) and (56).*

Proof. Formula (64) is deduced from Eq. (28), (50), and (55).

Remark 4.

(i) According to the Propositions 1, 2, and 4, the dimensionless velocity x_r of the Rayleigh waves is defined by either Eq. (34) or Eq. (64), depending on the values of δ_3. In particular, if $\delta_3=0$, x_r is calculated by Eq. (34), provided that Eq. (33) holds; if $\delta_3 \neq 0$, it is given by Eq. (64) provided that Eq. (45) is valid. It is stressed that the formulas (34) and (64) are valid for a general strain-energy function W.

(ii) The dimensionless Rayleigh wave velocity x_r is a continuous function of two dimensionless parameters δ_1 and δ_2 for the case $\delta_3=0$ (see Eq. (34)), and of three dimensionless parameters δ_1, δ_2, and δ_3 for the case $\delta_3 \neq 0$ (see Eq. (64)).

(iii) When the half-space is unstressed, according to Eq. (7): $\alpha=\beta=\gamma=\mu$, thus in view of Eq. (23) and $\sigma_2=0$ it follows

$$\delta_1 = \delta_2 = \delta_3 = 1 \tag{65}$$

Since $\delta_3=1 \neq 0$, x_r is given by Eq. (64), as remarked above. From Eqs. (48), (52), (56), (64), and (65) we obtain the exact value of the dimensionless Rayleigh wave velocity for the incompressible linear elastic solids (without prestresses), namely

$$x_{r0} = 1 - \eta_0^2 \tag{66}$$

where

$$\eta_0 = \left(\frac{26}{27} + \frac{2}{3}\sqrt{\frac{11}{3}} \right)^{1/3} - \frac{8}{9}\left(\frac{26}{27} + \frac{2}{3}\sqrt{\frac{11}{3}} \right)^{-1/3} - \frac{1}{3} \tag{67}$$

The result, Eqs. (66) and (67), was first obtained by Ogden and Pham [9] in 2004. By Eq. (67), the approximate value of η_0 is 0.2956 (see also Ref. [18]), so the approximate value of x_{r0} is 0.9126. This agrees with the classical result for the incompressible linear elasticity (see, e.g., Ref. [20]). An alternative expression for x_{r0} was found by Malischewsky [21] in 2000, namely

$$x_{r0} = \tfrac{2}{3}(4 + \sqrt[3]{-17 + 3\sqrt{33}} - \sqrt[3]{17 + 3\sqrt{33}}) \tag{68}$$

which yields a simpler representation of η_0^2

$$\eta_0^2 = 1 - \tfrac{2}{3}(4 + \sqrt[3]{-17 + 3\sqrt{33}} - \sqrt[3]{17 + 3\sqrt{33}}) \tag{69}$$

It is clear from Eqs. (34) and (64) that the Rayleigh wave velocity depends on the type of the strain-energy function W, in general. Interestingly there is a special case for which the formula for the Rayleigh wave velocity does not depend on the type of W. This is the case of hydrostatic stress (see Ref. [18]), when $\lambda_1=\lambda_2=\lambda_3=1$ and $\sigma_1=\sigma_2=\sigma_3=\sigma$. In this case, as indicated by Dowaikh and Ogden [18], $\alpha=\beta=\gamma=\mu$ thus $\delta_1=\delta_2=1$. It is not difficult to verify that in this case the dimensionless Rayleigh wave velocity x_r is defined by

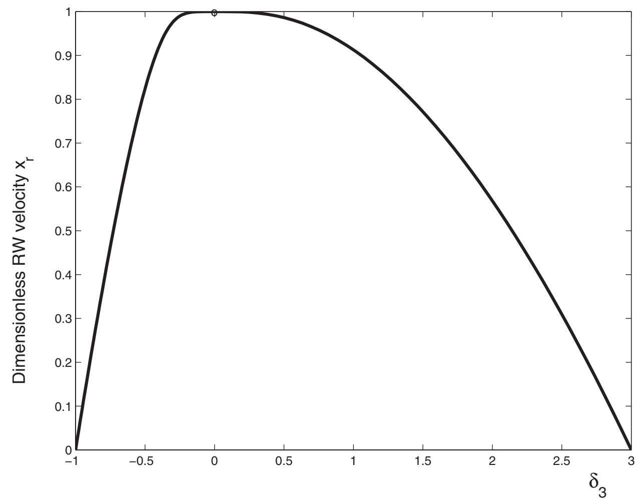

Fig. 1 Dependence of dimensionless Rayleigh velocity x_r on $\delta_3 \in (-1,3)$ for the case of hydrostatic stress

$$x_r = 1 - \left[\sqrt[3]{R + \sqrt{D}} - \frac{2}{9}\frac{(1 + 3\delta_3)}{\sqrt[3]{R + \sqrt{D}}} - \frac{1}{3} \right]^2, \quad -1 < \delta_3 < 3, \quad \delta_3 \neq 0 \tag{70}$$

where $\delta_3=1-\bar{\sigma}$, $\bar{\sigma}=\sigma/\mu$, and

$$R = (3\delta_3 + 1)(9\delta_3^2 - 3\delta_3 + 7)/54, \quad D = (\delta_3 + 1)^2(27\delta_3^2 + 16\delta_3 + 3)/108 \tag{71}$$

It is noted that x_r does not define at $\delta_3=0$ ($\bar{\sigma}=1$) because Eq. (33) is not valid for this case (noting $\delta_2=1$). Figure 1 shows the dependence of x_r on δ_3 in the interval $(-1,3)$. Note that x_r is ζ in Ref. [18].

Taking $\delta_3=1(\bar{\sigma}=0)$ in Eqs. (70) and (71), we again obtain the exact value of the dimensionless Rayleigh wave velocity for the incompressible linear elastic solids, which is defined by Eqs. (66) and (67).

4 Formulas for Particular Strain-Energy Functions

In this section we concretize the formulas (34) and (64) for some specific strain-energy functions, which were considered in Ref. [18]. For seeking simplicity, we confine ourself to the case of plane strain.

4.1 The Neo-Hookean Strain-Energy Function. For the neo-Hookean strain-energy function, we have (see Ref. [18])

$$W = \tfrac{1}{2}\mu(\lambda_1^2 + \lambda_1^{-2}\lambda_3^{-2} + \lambda_3^2 - 3) \tag{72}$$

It is noted that since $\lambda_1\lambda_2\lambda_3=1$, $\lambda_1^{-2}\lambda_3^{-2}=\lambda_2^2$. When the underlying deformation of the half-space corresponds to strain plane with $\lambda_3=1$, we write $\lambda_1=\lambda$, $\lambda_3=\lambda^{-1}$, and

$$W = \tfrac{1}{2}\mu(\lambda^2 + \lambda^{-2} - 2) \tag{73}$$

With the use of Eqs. (4)–(6), (13), and (73) we have

$$\alpha = \mu\lambda^2, \quad \beta = \frac{1}{2}\mu\left(\lambda^2 + \frac{1}{\lambda^2}\right), \quad \gamma = \frac{\mu}{\lambda^2} \tag{74}$$

Using Eqs. (23) and (74) provides

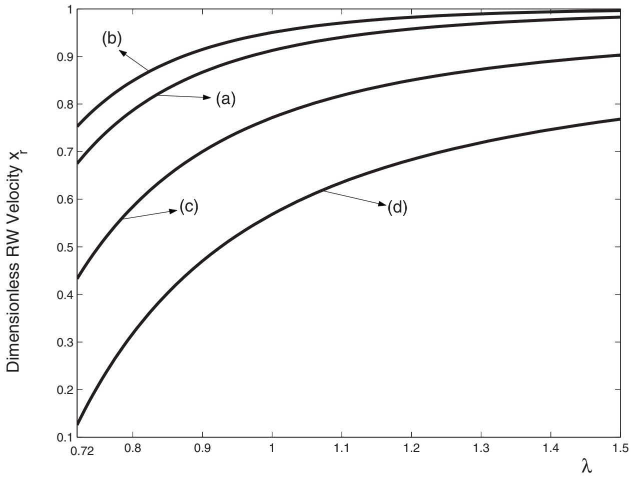

Fig. 2 Dependence of the dimensionless Rayleigh velocity x_r on $\lambda \in [0.72\ \ 1.5]$ for different given values of $\bar{\sigma}_2$: $\bar{\sigma}_2=0$ (line a), $\bar{\sigma}_2=0.2$ (line b), $\bar{\sigma}_2=-0.5$ (line c), and $\bar{\sigma}_2=-1$ (line d) for the case $\delta_3 \neq 0$; $W=\mu(\lambda^2+\lambda^{-2}-2)/2$

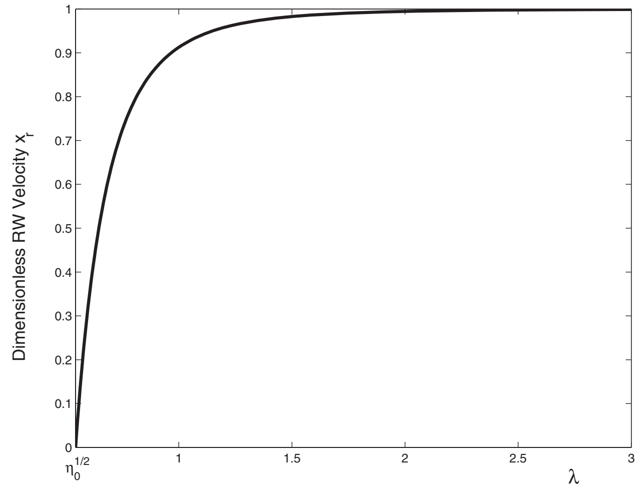

Fig. 3 Plot of x_r on $\lambda \in [\eta_0^{1/2} 3]$ for neo-Hookean strain-energy function and $\sigma_2=0$

$$\delta_1 = \frac{1}{\lambda^4}, \quad \delta_2 = \frac{1}{2}\left(1 + \frac{1}{\lambda^4}\right), \quad \delta_3 = \frac{1}{\lambda^4} - \frac{\bar{\sigma}_2}{\lambda^2}, \quad \bar{\sigma}_2 = \sigma_2/\mu \tag{75}$$

Since $2\delta_2 = 1 + 1/\lambda^4 > 1$ (noting that $\lambda > 0$), condition (33) is not satisfied, thus, x_r does not define at $\delta_3 = 0(\bar{\sigma}_2 = 1/\lambda^2)$. For the values of $\bar{\sigma}_2$ such that $\bar{\sigma}_2 \neq 1/\lambda^2(\delta_3 \neq 0)$, x_r is expressed by Eq. (64) provided that Eq. (45) is valid.

From Eqs. (48) and (75) it follows

$$a_0 = -\left(\frac{1}{\lambda^3} - \frac{\bar{\sigma}_2}{\lambda}\right)^2, \quad a_1 = \frac{3}{\lambda^4} - \frac{2\bar{\sigma}_2}{\lambda^2}, \quad a_2 = \frac{1}{\lambda^2} \tag{76}$$

Substituting Eq. (76) into Eqs. (52) and (56), and after some manipulation, we have

$$q^2 = \frac{2}{9\lambda^2}\left(3\bar{\sigma}_2 - \frac{4}{\lambda^2}\right)$$

$$R = \left(\frac{27\bar{\sigma}_2^2}{\lambda^2} - \frac{72\bar{\sigma}_2}{\lambda^4} + \frac{52}{\lambda^6}\right)/54$$

$$D = \left(\frac{176}{\lambda^{12}} - \frac{448\bar{\sigma}_2}{\lambda^{10}} + \frac{424\bar{\sigma}_2^2}{\lambda^8} - \frac{176\bar{\sigma}_2^3}{\lambda^6} + \frac{27\bar{\sigma}_2^4}{\lambda^4}\right)/108 \tag{77}$$

Finally, in view of Eqs. (64), (76), and (77), the dimensionless Rayleigh wave velocity x_r is defined by the following formula:

$$x_r = 1 - \left[\sqrt[3]{R + \sqrt{D}} + \frac{2}{9\lambda^2}\frac{(3\bar{\sigma}_2 - 4/\lambda^2)}{\sqrt[3]{R + \sqrt{D}}} - \frac{1}{3\lambda^2}\right]^2 \tag{78}$$

where R and D are given by Eq. (77). By Eqs. (45) and (75) it follows

$$\lambda^{-2} - \lambda^{-1} - 1 - \lambda < \bar{\sigma}_2 < \lambda^{-2} + \lambda^{-1} - 1 + \lambda \tag{79}$$

Figure 2 shows dependence of the dimensionless Rayleigh velocity x_r on $\lambda \in [0.72\ \ 1.5]$ for different given values of $\bar{\sigma}_2$ for the case $\delta_3 \neq 0$.

Now we turn our attention to a special case in which $\bar{\sigma}_2 = 0$. For this case, it follows from Eq. (77) that

$$R = \frac{26}{27\lambda^6}, \quad D = \frac{44}{27\lambda^{12}} \tag{80}$$

and by Eq. (78) we obtain

$$x_r = 1 - \frac{1}{\lambda^4}\eta_0^2, \quad \lambda > \eta_0^{1/2} \tag{81}$$

It is noted that this case was numerically examined by Dowaikh and Ogden [18]. Here the explicit expressions for the dimensionless Rayleigh wave velocity x_r is obtained. Figure 3 shows the plot of x_r as a function of λ, defined by Eq. (81).

4.2 The Varga Strain-Energy Function. The Varga strain-energy function is of the form (see Ref. [18])

$$W = 2\mu(\lambda_1 + \lambda_1^{-1}\lambda_3^{-1} + \lambda_3 - 3) \tag{82}$$

In the plane strain $\lambda_3 = 1$, so that

$$W = 2\mu(\lambda + \lambda^{-1} - 2) \tag{83}$$

Here we write $\lambda_1 = \lambda$.

From Eqs. (4)–(6), (13), and (83) we have

$$\alpha = \frac{2\mu\lambda^3}{\lambda^2 + 1}, \quad \beta = \frac{2\mu\lambda}{\lambda^2 + 1}, \quad \gamma = \frac{2\mu}{\lambda(\lambda^2 + 1)} \tag{84}$$

Using Eqs. (23) and (84) yields

$$\delta_1 = \frac{1}{\lambda^4}, \quad \delta_2 = \frac{1}{\lambda^2}, \quad \delta_3 = \frac{1}{\lambda^4} - \frac{(\lambda^2 + 1)\bar{\sigma}_2}{2\lambda^3} \tag{85}$$

(a) If $\delta_3 \neq 0$ and $F(1) > 0$, then x_r is given by Eq. (64). Using Eqs. (48) and (85) gives

$$a_0 = -\left(\frac{1}{\lambda^3} - \frac{(\lambda^2 + 1)\bar{\sigma}_2}{2\lambda^2}\right)^2, \quad a_1 = \frac{2}{\lambda^4} + \frac{2}{\lambda^2} - \frac{(\lambda^2 + 1)\bar{\sigma}_2}{\lambda^3} - 1, \quad a_2 = \frac{1}{\lambda^2} \tag{86}$$

Substituting Eq. (86) into Eqs. (52) and (56), and after some manipulation we have

$$q^2 = \frac{1}{3}\left(\frac{(\lambda^2 + 1)\bar{\sigma}_2}{\lambda^3} + 1 - \frac{2}{\lambda^2} - \frac{5}{3\lambda^4}\right) \tag{87}$$

$$R = \left(\frac{43}{\lambda^6} + \frac{18}{\lambda^4} - \frac{9}{\lambda^2} - \frac{36(\lambda^2 + 1)\bar{\sigma}_2}{\lambda^5} + \frac{27(\lambda^2 + 1)^2\bar{\sigma}_2^2}{4\lambda^4}\right)/54 \tag{88}$$

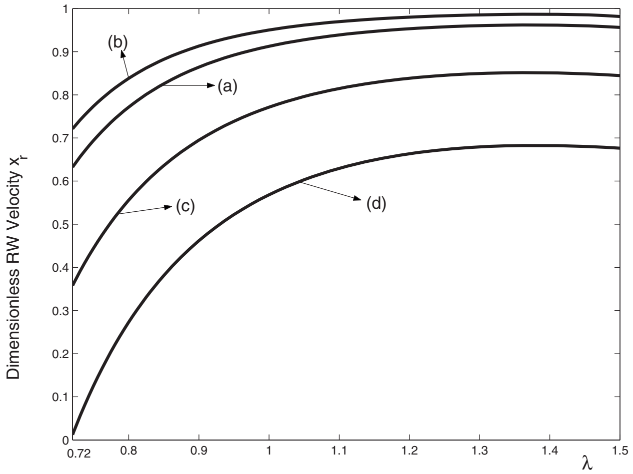

Fig. 4 Dependence of dimensionless Rayleigh velocity x_r on $\lambda \in [0.72\ \ 1.5]$ for different given values of $\bar{\sigma}_2$: $\bar{\sigma}_2=0$ (line a), $\bar{\sigma}_2=0.2$ (line b), $\bar{\sigma}_2=-0.5$ (line c), and $\bar{\sigma}_2=-1$ (line d) for the case $\delta_3 \neq 0$; $W=2\mu(\lambda+\lambda^{-1}-2)$

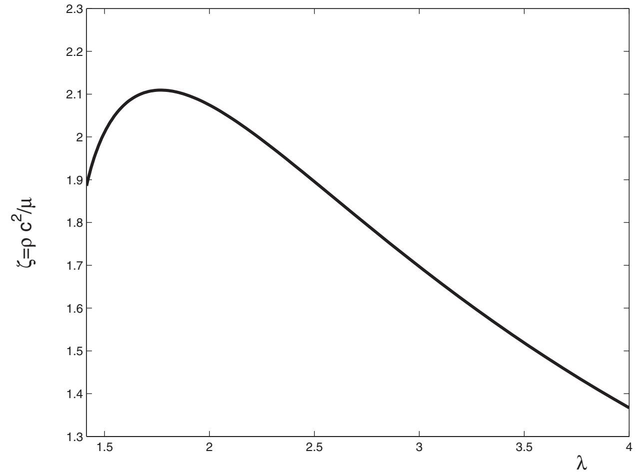

Fig. 5 Plot of $\zeta=\rho c^2/\mu$ as a function of $\lambda(\sqrt{2}<\lambda<4)$ for the case $\delta_3=0$ and the Varga strain-energy function

$$D=\left[\frac{87}{\lambda^{12}}+\frac{124}{\lambda^{10}}+\frac{30}{\lambda^{8}}-\frac{60}{\lambda^{6}}-\frac{25}{\lambda^{4}}+\frac{24}{\lambda^{2}}-4 -\frac{(\lambda^2+1)\bar{\sigma}_2}{2\lambda^3}\left(\frac{296}{\lambda^8}+\frac{256}{\lambda^6}-\frac{32}{\lambda^4}-\frac{96}{\lambda^2}+24\right) +\frac{(\lambda^2+1)^2\bar{\sigma}_2^2}{4\lambda^6}\left(\frac{358}{\lambda^4}+\frac{132}{\lambda^2}-66\right)-\frac{22(\lambda^2+1)^3\bar{\sigma}_2^3}{\lambda^9} +\frac{27(\lambda^2+1)^4\bar{\sigma}_2^4}{16\lambda^8}\right. \tag{89}$$

Finally, in view of Eqs. (64), (86), and (87), x_r is expressed by the following formula:

$$x_r=1-\left[\sqrt[3]{R+\sqrt{D}}+\frac{1}{9}\frac{3(\lambda^{-1}+\lambda^{-3})\bar{\sigma}_2+3-6\lambda^{-2}-5\lambda^{-4}}{\sqrt[3]{R+\sqrt{D}}}-\frac{1}{3\lambda^2}\right]^2 \tag{90}$$

where R and D are given by Eqs. (88) and (89). From Eqs. (45) and (85) it is deduced that

$$2\lambda^{-1}(1-3\lambda^2)/(\lambda^2+1)<\bar{\sigma}_2<2\lambda^{-1} \tag{91}$$

When $\bar{\sigma}_2=0$, Eqs. (88) and (89) simplify to

$$R=\left(\frac{43}{\lambda^6}+\frac{18}{\lambda^4}-\frac{9}{\lambda^2}\right)/54$$

$$D=\left(\frac{87}{\lambda^{12}}+\frac{124}{\lambda^{10}}+\frac{30}{\lambda^{8}}-\frac{60}{\lambda^{6}}-\frac{25}{\lambda^{4}}+\frac{24}{\lambda^{2}}-4\right)/108 \tag{92}$$

and x_r is given by

$$x_r=1-\left[\sqrt[3]{R+\sqrt{D}}+\frac{1}{9}\frac{(3-6\lambda^{-2}-5\lambda^{-4})}{\sqrt[3]{R+\sqrt{D}}}-\frac{1}{3\lambda^2}\right]^2,\quad \lambda>\frac{1}{\sqrt{3}} \tag{93}$$

in which R and D are calculated by Eq. (92).

Figure 4 shows dependence of the dimensionless Rayleigh velocity x_r on $\lambda \in [0.72\ \ 1.5]$ for different given values of $\bar{\sigma}_2$ for the case $\delta_3 \neq 0$.

(b) If $\delta_3=0(\bar{\sigma}_2=2/\lambda(\lambda^2+1))$ and Eq. (33) holds, then x_r is expressed by Eq. (34). In particular

$$x_r=1-\frac{1}{4\lambda^4}(\sqrt{1+4\lambda^2(\lambda^2-2)}-1)^2,\quad \lambda>\sqrt{2} \tag{94}$$

The dependence of x_r on $\lambda(\sqrt{2}<\lambda<4)$ for this case is shown in Fig. 5.

4.3 The $m=1/2$ Strain-Energy Function. The $m=1/2$ strain-energy function is of the form (see Ref. [18])

$$W=8\mu(\lambda_1^{1/2}+\lambda_1^{-1/2}\lambda_3^{-1/2}+\lambda_3^{1/2}-3) \tag{95}$$

In the plane strain $\lambda_3=1$, W becomes

$$W=8\mu(\lambda^{1/2}+\lambda^{-1/2}-2) \tag{96}$$

Here we write $\lambda_1=\lambda$.

From Eq. (4)–(6), (13), and (96) we have

$$\alpha=\frac{4\mu\lambda^4}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)},\quad \beta=\frac{\mu(-\lambda^4+2\lambda^3+2\lambda^2+2\lambda-1)}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)},\quad \gamma=\frac{\alpha}{\lambda^4} \tag{97}$$

Using Eqs. (23) and (97) yields

$$\delta_1=\frac{1}{\lambda^4},\quad \delta_2=\frac{1}{4}\left(-1+\frac{2}{\lambda}+\frac{2}{\lambda^2}+\frac{2}{\lambda^3}-\frac{1}{\lambda^4}\right),\quad \delta_3=\frac{1}{\lambda^4}-e,\quad e=\frac{\sigma_2}{\alpha} \tag{98}$$

(a) If $\delta_3 \neq 0$ and $F(1)>0$, then x_r is given by Eq. (64). Using Eqs. (48) and (98) gives

$$a_0=-\left(\frac{1}{\lambda^3}-\lambda e\right)^2,\quad a_1=\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{1}{\lambda^3}+\frac{3}{2\lambda^4}-2e-\frac{3}{2},\quad a_2=\frac{1}{\lambda^2} \tag{99}$$

Substituting Eq. (99) into Eqs. (52) and (56), and after some manipulation, we have

$$q^2=\frac{1}{9}\left(6e+\frac{9}{2}-\frac{3}{\lambda}-\frac{3}{\lambda^2}-\frac{3}{\lambda^3}-\frac{7}{2\lambda^4}\right)$$

$$R=\left(\frac{77}{2\lambda^6}+\frac{9}{\lambda^5}+\frac{9}{\lambda^4}+\frac{9}{\lambda^3}-\frac{9(8e+3/2)}{\lambda^2}+27\lambda^2e^2\right)/54$$

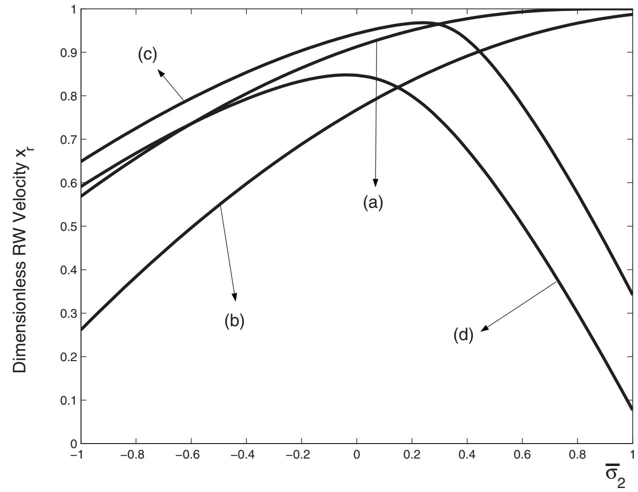

Fig. 6 Dependence of dimensionless Rayleigh velocity x_r on $\bar{\sigma}_2 \in [-1\ \ 1]$ for different given values of λ: λ=1 (line a), λ=0.8 (line b), λ=1.5 (line c), and λ=1.7 (line d) for the case $\delta_3 \neq 0$; $W=8\mu(\lambda^{1/2}+\lambda^{-1/2}-2)$

$$D = \lambda^4\Bigg[\frac{245}{\lambda^{16}} + \frac{168}{\lambda^{15}} + \frac{236}{\lambda^{14}} + \frac{320}{\lambda^{13}} - \frac{952e}{\lambda^{12}} + \frac{(28-416e)}{\lambda^{11}} - \frac{(96+512e)}{\lambda^{10}} - \frac{(252+608e)}{\lambda^{9}} + \frac{(-15+336e+1300e^2)}{\lambda^8} + \frac{(-20+96e+264e^2)}{\lambda^7} + \frac{(36+192e+264e^2)}{\lambda^6} + \frac{(108+288e+264e^2)}{\lambda^5} - \frac{(54+216e+396e^2+704e^3)}{\lambda^4} + 108e^4\Bigg]/432 \quad (100)$$

Finally, in view of Eqs. (64) and (100), x_r is expressed by the following formula:

$$x_r = 1 - \left[\sqrt[3]{R+\sqrt{D}} + \frac{1}{9}\frac{(6e+9/2-3/\lambda-3/\lambda^2-3/\lambda^3-3.5/\lambda^4)}{\sqrt[3]{R+\sqrt{D}}} - \frac{1}{3\lambda^2}\right]^2 \quad (101)$$

where R and D are given by Eq. (100). By Eqs. (47), (97), and the fourth of Eq. (98) e is subjected to

$$\frac{1-\lambda}{\lambda^4} - \frac{(1+\lambda)}{\sqrt{2}\lambda^3}\sqrt{-\lambda^2+4\lambda-1} < e < \frac{1-\lambda}{\lambda^4} + \frac{(1+\lambda)}{\sqrt{2}\lambda^3}\sqrt{-\lambda^2+4\lambda-1} \quad (102)$$

or the bounds on $\bar{\sigma}_2=\sigma_2/\mu$ are

$$\frac{4(1-\lambda)}{\sqrt{\lambda}(\lambda^2+1)} \pm \frac{2\sqrt{2\lambda}}{(\lambda^2+1)}\sqrt{-\lambda^2+4\lambda-1} \quad (103)$$

For $\sigma_2=0$, λ is subject to

$$-\lambda^3+3\lambda^2+2\lambda-2>0 \quad (104)$$

Figure 6 shows dependence of x_r on $\bar{\sigma}_2 \in [-1\ \ 1]$ for different given values of λ for the case $\delta_3 \neq 0$.

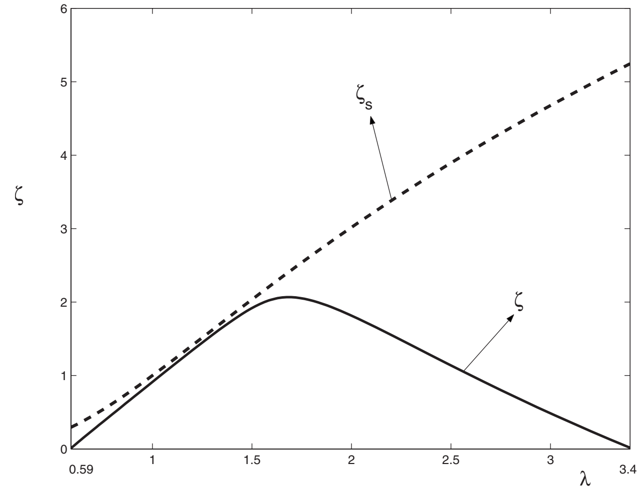

Fig. 7 Plots of $\zeta=\rho c^2/\mu$ (solid line) and $\zeta_s=\alpha/\mu$ (dashed line) as functions of λ for $\sigma_2=0$ and $m=1/2$ strain-energy function

(b) When $\delta_3=0(e=1/\lambda^4)$ and Eq. (33) holds, x_r is expressed by Eq. (34), in particular

$$x_r = 1 - \frac{1}{4\lambda^4}(\sqrt{6\lambda^4-4\lambda^3-4\lambda^2-4\lambda+3}-1)^2 \quad (105)$$

in which λ must satisfy the following two inequalities:

$$-1+2/\lambda+4/\lambda^2+2/\lambda^3-1/\lambda^4>0>-3+2/\lambda+2/\lambda^2+2/\lambda^3-1/\lambda^4$$

Plots of $\zeta=\rho c^2/\mu$ and $\zeta_s=\alpha/\mu$ as functions of λ for $\sigma_2=0$ are shown in Fig. 7.

Finally, we note that formulas (78), (90), and (101) all lead to

$$x_r = 1-\eta_0^2 \quad \text{at} \quad \lambda=1, \quad \sigma_2=0 \quad (106)$$

where η_0 defined by Eq. (67). This means these formulas all recover the (exact) value of the dimensionless speed of the Rayleigh wave in incompressible isotropic elastic materials (without prestrain).

5 Conclusions and Remarks

In this paper, formulas for the Rayleigh wave velocity in incompressible isotropic solids subject to uniform initial deformation are derived using the theory of cubic equation. They have a simple algebraic form, valid for any range of prestrain and hold for a general strain-energy function. The Rayleigh wave velocity is expressed by two different formulas depending on that $\delta_3=0$ or $\delta_3 \neq 0$. These formulas are concretized for a number of forms of strain-energy function, and the obtained formulas express the Rayleigh wave velocity as totally explicit continuous functions of the principle stretches of the deformation λ_i and the stress σ_2. For the case of hydrostatic stress, the Rayleigh wave velocity is expressed by a simple formula that does not include the strain-energy function.

The obtained formulas will provide a good tool for the nondestructive evaluation of prestresses of structures. In relation to the use of these formulas, we emphasize the following points.

(i) By $x_r^{(ik)}(i,k=1,2,3,i\neq k)$ we denote the velocity of the Rayleigh wave propagating in the x_i-direction and decaying in the x_k-direction, then $x_r^{(ik)}$ is defined by formulas similar to Eqs. (34) and (64). For example, if $\delta_3^{(32)} \neq 0$ (i.e.,

$\sigma_2 \neq \gamma^{(32)} = B_{2323}$) then $x_r^{(32)}$ defined by formula (64), along with Eqs. (23), (48), (52), and (56), in which

$$\alpha^{(32)} = B_{3232}, \quad \gamma^{(32)} = B_{2323}, \quad 2\beta^{(32)} = B_{3333} + B_{2222} - 2B_{3322} - 2B_{3223} \tag{107}$$

and $\gamma_*^{(32)} = \gamma^{(32)} - \sigma_2$. Formula (107) is derived from Eq. (13) in which the index 1 of B_{ijkl} is replaced by 3. Note that, for a given material $x_r^{(ik)}$ is a function of two of the three principal stretches (because $\lambda_1\lambda_2\lambda_3 = 1$) and σ_k, i.e.,

$$x_r^{(ik)} = f^{(ik)}(\lambda_1, \lambda_2, \sigma_k) \quad (i,k = 1,2,3, i \neq k) \tag{108}$$

(ii) If $\sigma_2 = 0$ and $x_r^{(12)}$, $x_r^{(32)}$ are known (by laser techniques, for example), then λ_1, λ_2 are determined from two following equations:

$$f^{(12)}(\lambda_1, \lambda_2, 0) = x_r^{(12)}, \; f^{(32)}(\lambda_1, \lambda_2, 0) = x_r^{(32)} \tag{109}$$

and then σ_1 and σ_3 are calculated from

$$\sigma_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2}, \quad \sigma_3 = \lambda_3 \frac{\partial W}{\partial \lambda_3} - \lambda_2 \frac{\partial W}{\partial \lambda_2} \tag{110}$$

which are originated from Eq. (9). Note that from Eq. (9) it follows

$$\sigma_1 - \sigma_2 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2}$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_3 \frac{\partial W}{\partial \lambda_3}$$

$$\sigma_3 - \sigma_1 = \lambda_3 \frac{\partial W}{\partial \lambda_3} - \lambda_1 \frac{\partial W}{\partial \lambda_1} \tag{111}$$

Also note that when $\sigma_2 = 0$, $\delta_3^{(12)} \neq 0$, and $\delta_3^{(32)} \neq 0$, therefore, $x_r^{(12)}$ and $x_r^{(32)}$ are defined by Eq. (64). A similar situation will be met when $\sigma_1 = 0$ or $\sigma_3 = 0$.

When $\sigma_k \neq 0$, $k = 1,2,3$, in order to determine λ_1, λ_2, σ_1, σ_2, and σ_3, we have to use five equations, two of which come from Eq. (111), for example

$$\lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2} = \sigma_1 - \sigma_2, \quad \lambda_3 \frac{\partial W}{\partial \lambda_3} - \lambda_2 \frac{\partial W}{\partial \lambda_2} = \sigma_3 - \sigma_2 \tag{112}$$

and the others are originated from the formulas of $x_r^{(ik)}$, for instance

$$f^{(12)}(\lambda_1, \lambda_2, \sigma_2) = x_r^{(12)}, \quad f^{(32)}(\lambda_1, \lambda_2, \sigma_2) = x_r^{(32)}, \quad f^{(21)}(\lambda_1, \lambda_2, \sigma_1) = x_r^{(21)} \tag{113}$$

Here $x_r^{(12)}$, $x_r^{(32)}$, and $x_r^{(21)}$ are known (by measurement techniques).

(iii) If from two equations, Eq. (112) and $\lambda_1\lambda_2\lambda_3 = 1$, we can obtain analytical expressions $\lambda_1 = \lambda_1(\sigma_1, \sigma_2, \sigma_3)$ and $\lambda_2 = \lambda_2(\sigma_1, \sigma_2, \sigma_3)$, then by introducing them into Eq. (108), $x_r^{(ik)}$ is expressed as a function of the prestresses $\sigma_k, k = 1,2,3$. However, such an analytical inversion of Eq. (112) and $\lambda_1\lambda_2\lambda_3 = 1$ is not always possible (see also Ref. [22], p. 150).

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